CONJUGATE STATIONARY PROBLEM OF HEAT TRANSFER WITH A MOVING FLUID IN A SEMI-INFINITE TUBE ALLOWING FOR VISCOUS DISSIPATION

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Heat transfer in a circular tube in the presence of laminar forced convection is investigated with allowance for thermal interaction at the solid-fluid boundary.

We consider the stationary problem of heat transfer for laminar forced convection in a circular tube. This problem was first investigated at the end of the nineteenth century in connection with the solution of the so-called Graetz problem. Since then the problem has been solved repeatedly for different boundary conditions and by different methods [1–4].

Usually, in solving problems of heat transfer between a solid body and a fluid flow the conditions at the inner surface of the body are assumed given.

In the presence of intense heat transfer, for example, this assumption is not satisfactory, since it does not account for the thermal interaction between solid and fluid. The temperature or flux at the inner surface cannot be given a priori, but should be obtained from a joint solution of the equations of heat propagation in the fluid and in the solid. It is then necessary to solve the so-called conjugate problem of heat transfer.

Certain specific examples of this problem were formulated and solved in [5, 6].

We obtain an exact solution of the conjugate problem of heat transfer in a semi-infinite circular tube of finite thickness filled with a moving incompressible fluid, for a steady-state Poiseuille velocity distribution with allowance for mechanical energy dissipation.

§1. Mathematically, the problem reduces to the solution in dimensionless variables of the equation for the fluid

$$(1-\rho^2)\frac{\partial\theta}{\partial\xi} = \frac{1}{\mathrm{Pe}^2}\frac{\partial^2\theta}{\partial\xi^2} + \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} + \frac{\partial^2\theta}{\partial\rho^2} + H_1\rho^2, \quad 0 \leqslant \rho \leqslant 1, \quad 0 \leqslant \xi < \infty, \tag{1}$$

with boundary conditions

$$\theta|_{\xi=0} = 0, \quad \theta|_{\xi\to\infty} = \frac{H_1}{16}(1-\rho^4) + 1, \tag{2}$$

$$\left.\frac{\partial\theta}{\partial\rho}\right|_{\rho=0} = 0, \quad \theta|_{\rho=1} = \frac{\chi(\xi)}{\chi(\infty)}, \tag{3}$$

and the equation for the solid

$$\frac{\partial^2\theta_e}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\theta_e}{\partial\rho} + \frac{\partial^2\theta_e}{\partial\xi^2} = 0, \quad 1 \leqslant \rho \leqslant \frac{R_1}{R}, \quad 0 \leqslant \xi < \infty, \tag{4}$$

with boundary conditions

$$\theta_e|_{\xi=0} = 0, \quad \theta_e|_{\xi\to\infty} = -K_\lambda\frac{H_1}{4}\ln\rho + 1, \tag{5}$$

$$\theta_e|_{\rho=1} = \frac{\chi(\xi)}{\chi(\infty)}, \quad \theta_e|_{\rho=\frac{R_1}{R}} = \frac{\psi(\xi)}{\chi(\infty)}, \tag{6}$$

while at the solid-fluid boundary we have

$$\theta|_{\rho=1} = \theta_e|_{\rho=1} = \frac{\chi(\xi)}{\chi(\infty)}, \tag{7}$$

$$K_\lambda\frac{\partial\theta}{\partial\rho}\Big|_{\rho=1} = \left.\frac{\partial\theta_e}{\partial\rho}\right|_{\rho=1}. \tag{8}$$

The conditions for $\xi\to\infty$ in (2) and (5)

$$\chi(\infty) = T\Big|_{\substack{r=1\\ z\to\infty}} = \psi(\infty) + K_\lambda\ln\frac{R_1}{R}\,\frac{H}{4a}R^4 \tag{9}$$

were obtained from the solution of the problem for $z\to\infty$.

§2. Consider first the boundary value problem (1)–(3). Usually the Pe number is very large and Eq. (1) becomes

$$(1-\rho^2)\frac{\partial\theta}{\partial\xi} = \frac{1}{\rho}\frac{\partial\theta}{\partial\rho} + \frac{\partial^2\theta}{\partial\rho^2} + H_1\rho^2, \quad 0 \leqslant \rho \leqslant 1, \quad 0 \leqslant \xi < \infty. \tag{10}$$

We seek a solution of (10) with (2)–(3) in the form

$$\theta = \theta_1 + \theta_2, \tag{11}$$

where θ_1 is the solution of (10) satisfying the boundary conditions

$$\theta_1|_{\xi=0} = 0, \quad \theta_1|_{\xi\to\infty} = \frac{H_1}{16}(1-\rho^4), \tag{12}$$

$$\left.\frac{\partial\theta_1}{\partial\rho}\right|_{\rho=0} = 0, \quad \theta_1|_{\rho=1} = 0, \tag{13}$$

and θ_2 is the solution of

$$(1-\rho^2)\frac{\partial u}{\partial\xi} = \frac{1}{\rho}\frac{\partial u}{\partial\rho} + \frac{\partial^2 u}{\partial\rho^2}, \tag{14}$$

satisfying the conditions

$$\theta_2|_{\xi=0} = 0, \quad \theta_2|_{\xi\to\infty} = 1, \tag{15}$$

$$\left.\frac{\partial\theta_2}{\partial\rho}\right|_{\rho=0} = 0, \quad \theta_2|_{\rho=1} = \frac{\chi(\xi)}{\chi(\infty)}. \tag{16}$$

We seek a solution of (10), (12)–(13) as a sum of two functions,

$$\theta_1 = \frac{H_1}{16}(1-\rho^4) + \theta_3, \tag{17}$$

where θ_3 satisfies an equation of the form (14) with boundary conditions

$$\theta_3|_{\xi=0} = -\frac{H_1}{16}(1-\rho^4), \quad \theta_3|_{\xi\to\infty} = 0, \tag{18}$$

$$\left.\frac{\partial\theta_3}{\partial\rho}\right|_{\rho=0} = 0, \quad \theta_3|_{\rho=1} = 0. \tag{19}$$

Obviously, θ_3 is of the form

$$\theta_3 = \sum_{n=1}^{\infty} B_n R_n(\rho)\exp[-b_n\xi], \tag{20}$$

where the $R_n(\rho)$ satisfy

$$R_n''(\rho) + \frac{1}{\rho}R_n'(\rho) + [P_n^2(1-\rho^2)]\,R_n(\rho) = 0 \tag{21}$$

and the boundary conditions

$$\left.\frac{dR_n(\rho)}{d\rho}\right|_{\rho=0} = 0, \quad R_n(\rho)|_{\rho=1} = 0, \tag{22}$$

where

$$P_n^2 = b_n. \tag{23}$$

It is easy to show that the solution of (21)–(22) is

$$R_n(\rho) = \exp\left[-\frac{P_n}{2}\rho^2\right]{}_1F_1(a,\ 1,\ P_n\rho^2), \tag{24}$$

where $a = 1/2 - P_n/4$; ${}_1F_1(a, 1, P_n\rho^2)$ is a confluent hypergeometric function.

From (22) with $\rho = 1$ we obtain the characteristic equation for P_n, i.e.,

$${}_1F_1(a,\ 1,\ P_n) = 0, \tag{25}$$

(the roots of (25) are given in [7]).

The constants B_n are determined by substituting (20) into (18) with $\xi = 0$ taking into account the orthogonality of $R_n(\rho)$ on the interval [0, 1]

$$B_n = -\frac{H_1\int_0^1 (1-\rho_1^4)\rho_1(1-\rho_1^2)R_n(\rho_1)\,d\rho_1}{16\int_0^1 \rho_1(1-\rho_1^2)R_n^2(\rho_1)\,d\rho_1} =$$

$$= -\frac{H_1P_n\int_0^1 (1-\rho_1^4)\rho_1(1-\rho_1^2)R_n(\rho_1)\,d\rho_1}{8\left[\frac{\partial R_n}{\partial P_n}\frac{\partial R_n}{\partial\rho}\right]_{\rho=1}}. \tag{26}$$

We proceed to determine θ_2.

Applying the superposition principle, we obtain

$$\theta_2 = -\sum_{n=1}^{\infty} A_nR_n(\rho)b_n\int_0^z \exp[-b_n(\xi-\eta)]\frac{\chi(\eta)}{\chi(\infty)}\,d\eta =$$

$$= \frac{\chi(\xi)}{\chi(\infty)} + \sum_{n=1}^{\infty} A_nR_n(\rho)\exp[-b_n\xi]\frac{\chi(0)}{\chi(\infty)} +$$

$$+ \sum_{n=1}^{\infty} A_nR_n(\rho)\int_0^{\xi}\exp[-b_n(\xi-\eta)]\frac{\chi'(\eta)}{\chi(\infty)}\,d\eta,$$

$$A_n = -\frac{\int_0^1 R_n(\rho_1)\rho_1(1-\rho_1^2)\,d\rho_1}{\int_0^1 \rho_1(1-\rho_1^2)R_n^2(\rho_1)\,d\rho_1} = \frac{2}{P_n\left(\frac{\partial R_n}{\partial P_n}\right)_{\rho=1}}. \tag{27}$$

Considering (11), (17), (20), (27), we obtain the solution of (1)–(3):

$$\theta = \frac{H_1}{16}(1-\rho^4) + \sum_{n=1}^{\infty} B_nR_n(\rho)\exp[-b_n\xi] -$$

$$-\sum_{n=1}^{\infty} A_nR_n(\rho)\int_0^{\xi}\exp[-b_n(\xi-\eta)]\frac{\chi(\eta)}{\chi(\infty)}\,d\eta. \tag{28}$$

§3. We obtain the solution of problem (4)–(6) using the generalized Fourier sine transformation [8]. The transform is

$$\theta_{es}(\alpha,\rho) = \frac{1-K_\lambda\frac{H_1}{4}\ln\rho}{\alpha} - \alpha\int_0^{R_1/R} G(\rho,\rho_1,\alpha)f_2(\rho_1)\,d\rho_1 + \left[\frac{\chi_s}{\chi(\infty)} - \frac{1}{\alpha}\right]\times$$

$$\times\frac{K_0\left(\alpha\frac{R_1}{R}\right)I_0(\alpha\rho) - I_0\left(\alpha\frac{R_1}{R}\right)K_0(\alpha\rho)}{I_0(\alpha)K_0\left(\alpha\frac{R_1}{R}\right) - I_0\left(\alpha\frac{R_1}{R}\right)K_0(\alpha)}, \tag{29}$$

where

$$f_2(\rho) = \frac{H_1K_\lambda}{4}\ln\frac{\rho}{R_1} - \frac{\psi(\infty)}{\chi(\infty)},$$

$$G(\rho,\rho_1,\alpha) = \begin{cases} \dfrac{\alpha\left[K_0(\alpha\rho_1)I_0\left(\alpha\frac{R_1}{R}\right) - K_0\left(\alpha\frac{R_1}{R}\right)I_0(\alpha\rho_1)\right]}{I_0\left(\alpha\frac{R_1}{R}\right)}\times I_0(\alpha\rho), & 1\leqslant\rho\leqslant\rho_1, \\[2ex] \dfrac{\alpha\left[K_0(\alpha\rho)I_0\left(\alpha\frac{R_1}{R}\right) - K_0\left(\alpha\frac{R_1}{R}\right)I_0(\alpha\rho)\right]}{I_0\left(\alpha\frac{R_1}{R}\right)}\times I_0(\alpha\rho_1), & \rho_1\leqslant\rho\leqslant\frac{R_1}{R}. \end{cases}$$

Inverse transforming in (29), we obtain

$$\theta_e = 1 - K_\lambda\frac{H_1}{4}\ln\rho + \lim_{\sigma\to 0}\frac{1}{\pi}\left\{\int_0^{\infty}\left[\frac{\chi_s(\alpha)}{\chi(\infty)} - \frac{1}{\alpha}\right]\times\right.$$

$$\times\frac{\left[K_0\left(\alpha\frac{R_1}{R}\right)I_0(\alpha\rho) - I_0\left(\alpha\frac{R_1}{R}\right)K_0(\alpha\rho)\right]}{I_0(\alpha)K_0\left(\alpha\frac{R_1}{R}\right) - I_0\left(\alpha\frac{R_1}{R}\right)K_0(\alpha)}\times$$

$$\times \exp[-\sigma\alpha]\sin\alpha\xi\, d\alpha - \int_0^\infty \exp[-\sigma\alpha]\sin\alpha\xi\, d\alpha \times$$

$$\times \int_1^{R_1/R} G(\rho, \rho_1, \alpha)\left[\frac{H_1K_\lambda}{4}\ln\frac{\rho}{R_1} - \frac{\psi(\infty)}{\chi(\infty)}\right] d\rho_1 \Big\}. \quad (30)$$

§4. Now we find the unknown function $\chi(\xi)$. Applying the generalized Fourier sine transformation to (28) and substituting the result obtained and (29) into the transformed condition (8), we obtain

$$\chi_s(\alpha)\left[\sum_{n=1}^\infty \frac{A_nR_n'(1)b_n^2}{b_n^2+\alpha^2} + K_\lambda^{-1}\Phi_1(1, \alpha)\right] -$$

$$-\frac{2}{\pi}\sum_{n=1}^\infty \frac{A_nR_n'(1)\alpha}{b_n^2+\alpha^2}\int_0^\infty \frac{Y\chi_s(Y)}{Y^2-\alpha^2}\,dY =$$

$$= -K_\lambda^{-1}\Phi_2(1, \alpha)\chi(\infty) + \left[-\frac{H_1}{4\alpha} + \right.$$

$$\left. + \sum_{n=1}^\infty \frac{B_nR_n'(1)\alpha}{b_n^2+\alpha^2}\right]\chi(\infty),$$

$$\Phi_1(1, \alpha) =$$

$$= \frac{\alpha\left[K_0\left(\alpha\frac{R_1}{R}\right)I_1(\alpha) + I_0\left(\alpha\frac{R_1}{R}\right)K_1(\alpha)\right]}{I_0(\alpha)K_0\left(\alpha\frac{R_1}{R}\right) - I_0\left(\alpha\frac{R_1}{R}\right)K_0(\alpha)},$$

$$\Phi_2(1, \alpha) = -\left\{\frac{K_\lambda H_1}{4\alpha} + \right.$$

$$+ \frac{K_0\left(\alpha\frac{R_1}{R}\right)I_1(\alpha) + I_0\left(\alpha\frac{R_1}{R}\right)K_1(\alpha)}{I_0(\alpha)K_0\left(\alpha\frac{R_1}{R}\right) - K_0(\alpha)I_0\left(\alpha\frac{R_1}{R}\right)} +$$

$$+\alpha\int_1^{R_1/R} G'(1, \rho_1, \alpha)\left[\frac{HK_\lambda \ln\frac{\rho_1}{R}}{4} - \frac{\psi(\infty)}{\chi(\infty)}\right] d\rho_1. \quad (31)$$

In (31) we make the change of variables

$$t = \alpha^2, \quad \tau = Y^2, \quad \chi_s(\alpha) = \varphi(t); \quad (32)$$

then

$$\varphi(t)a(t) - b(t)\int_0^\infty \frac{\varphi(\tau)}{\tau - t}\,d\tau = f(t), \quad (33)$$

where

$$a(t) = \sum_{n=1}^\infty \frac{A_nR_n'(1)b_n^2}{b_n^2+t} + K_\lambda^{-1}\Phi_1(1, \sqrt{t}),$$

$$b(t) = \frac{1}{\pi}\sum_{n=1}^\infty \frac{A_nR_n'(1)\sqrt{t}}{b_n^2+t},$$

$$f(t) = \left[-K_\lambda^{-1}\Phi_2(1, \sqrt{t}) - \frac{H_1}{4\sqrt{t}} + \right.$$

$$\left. + \sum_{n=1}^\infty \frac{B_nR_n'(1)\sqrt{t}}{b_n^2+t}\right]\chi(\infty).$$

Equation (33) is a singular integral equation with a Cauchy kernel [9].

§5. To solve (34) we use the idea of analytic continuation in the complex domain.

Equation (33) is reduced to a Riemann problem with discontinuous coefficients.

Introducing the piecewise-analytic function

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}\,d\tau, \quad (34)$$

where the contour L is the positive part of the real axis, and using the Sokhotskii–Premelj formula, we obtain

$$\varphi(t) = \Phi^+(t) - \Phi^-(t), \quad (35)$$

$$\frac{1}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - z}\,d\tau = \Phi^+(t) + \Phi^-(t). \quad (36)$$

Substituting (35)–(36) into (33), we have

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad (37)$$

where

$$G(t) = \frac{a(t) + b(t)\pi i}{a(t) - b(t)\pi i}, \quad g(t) = \frac{f(t)}{a(t) - b(t)\pi i}.$$

A solution of the inhomogeneous Riemann problem (37) can be easily obtained by taking the index of the problem $\nu = 0$ (the proof and the detailed solution of this assertion will be presented elsewhere). Then

$$\Phi^\pm(z) = X^\pm(z)\Psi^\pm(z),$$

where

$$X^+(z) = \exp[\Gamma^+(z)], \quad X^-(z) = \exp[\Gamma^-(z)], \quad (38)$$

$$\Gamma(z) = \frac{1}{2\pi i}\int_L \frac{\ln\bar{G}(\tau)}{\tau - z}\,d\tau, \quad (39)$$

$$\Psi(z) = \frac{1}{2\pi i}\int_L \frac{\bar{g}(\tau)}{X^+(\tau)}\frac{d\tau}{\tau - z},$$

$$\bar{G}(t) = \begin{cases} G(t), & 0 < t < \infty, \\ 1, & -\infty < t < 0; \end{cases}$$

$$\bar{g}(t) = \begin{cases} g(t), & 0 < t < \infty, \\ 0, & -\infty < t < 0. \end{cases} \quad (40)$$

Using the Sokhotskii–Plemelj formula and (38)–(40), we obtain

$$\Phi^+(t) = X^+(t)\left[\frac{1}{2}\frac{\bar{g}(t)}{X^+(t)} + \Psi(t)\right], \quad (41)$$

$$\Phi^-(t) = X^-(t)\left[-\frac{1}{2}\frac{\bar{g}(t)}{X^+(t)} + \Psi(t)\right], \quad (42)$$

where

$$X^+(t) = \sqrt{\bar{G}(t)}\exp[\Gamma(t)],$$

$$X^-(t) = \frac{1}{\sqrt{\bar{G}(t)}}\exp[\Gamma(t)],$$

$$\Gamma(t)=\frac{1}{2\pi i}\int_L \frac{\ln \bar{G}(\tau)}{\tau-t}\,d\tau,$$

$$\Psi(t)=\frac{1}{2\pi i}\int_L \frac{\bar{g}(\tau)}{X^+(\tau)}\,\frac{d\tau}{\tau-t}.$$

Substituting (41)–(42) into (35)

$$\varphi(t)=\frac{1}{2}g(t)\left[1+\frac{1}{\bar{G}(t)}\right]+$$

$$+X^+(t)\left[1-\frac{1}{\bar{G}(t)}\right]\Psi(t). \quad (43)$$

From (32) and (43), we derive the transform of the unknown function

$$\chi_s(\alpha)=\frac{1}{2}g(\alpha^2)\left[1+\frac{1}{G(\alpha^2)}\right]+\sqrt{G(\alpha^2)}\times$$

$$\times\exp\left[\Gamma(\alpha^2)\right]\left[1-\frac{1}{G(\alpha^2)}\right]\frac{1}{\pi i}\times$$

$$\times\int_0^\infty \frac{g(Y^2)\,Y dY}{\sqrt{G(Y^2)}\exp\left[\Gamma(Y^2)\right](Y^2-\alpha^2)}. \quad (44)$$

The unknown function $\chi(\xi)$ is given by

$$\chi(\xi)=\frac{1}{\pi}\lim_{\sigma\to 0}\int_0^\infty \chi_s(\alpha)\exp[-\sigma\eta]\sin\alpha\xi\,d\alpha. \quad (45)$$

The unknown temperatures θ and θ_e are obtained by substituting (45) and (44) into (28) and (30), respectively.

NOTATION

T is the temperature of the fluid; T_e is the temperature of the solid; $\psi(z)$ is the temperature at the outer surface of the tube; $\theta = T/\chi(\infty)$ is the dimensionless temperature of the fluid; $\theta_e = T_e/\chi(\infty)$ is the dimensionless temperature of the solid; r and z are space coordinates; $\xi = z/\mathrm{Pe}R$, $\rho = r/R$ are dimensionless space coordinates; R is the tube radius; $R_1 - R$ is the thickness of the solid; $\mathrm{Pe} = v_0 R/a$; $v_0 = 2v_1$; v_1 is the mean velocity; $\mathrm{Pr} = \nu/a$ is the Prandtl number; C is the specific heat; I is the mechanical equivalent of heat; $H_1 = 16\mathrm{Pr}\, v_1^2/CI\chi(\infty)$; $K_\lambda = \lambda/\lambda_e$; λ is the thermal conductivity of the fluid; λ_e is the thermal conductivity of the solid; $H = 16\nu v_1^2/CIR^4$.

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